Econometrics

Review of Conditional Probabilities

Paulo Fagandini
Marlon Francisco

Spring 2024-25

Random Variables

- X is a random variable if it takes on numerical values and has an outcome that is determined by an experiment
  - A discrete random variable can take on only a finite or countably infinite number of values
  - A continuous random variable can take on any real value with zero probability
- A probability distribution is associated with each random variable
- Examples:
  - Coin-flipping example: number of heads appearing in 10 flips of a coin
  - Height of a selected student

The Expected Value

- If X is a random variable, the expected value is a weighted average of all possible values of X
- The probability density function determines the weights
- Let $f(x)$ denote the probability density function of X and X a discrete random variable taking on a finite number of values $\{x_1, ..., x_n\}$

$$\mu(X) = E(X) \equiv \sum_{i=1}^{n} x_i f(x_i)$$

- Easily computed given the values of the pdf at each possible outcome of X

The Expected Value: Properties

- **E.1** For any constant c: $E(c) = c$
- **E.2** $E(E(X)) = E(X) = \mu(X)$
- **E.3** For any constants a and b: $E(aX + b) = aE(X) + b$
- **E.4** $E(X + Y) = E(X) + E(Y)$
- **E.5** $E(X - Y) = E(X) - E(Y)$
- **E.6** $E(X - \mu(X)) = 0$
- **E.7** $E((aX)^2) = a^2 E(X^2)$
- **E.8** If (and only if) X and Y are independent, then

$$E(XY) = E(X)E(Y)$$

The Expected Value: Properties

- **E.9** If $\{a_1, a_2, ...a_n\}$ are constants and $X_1, ..., X_n$ are random variables, then:

  $E(a_1X_1 + a_2X_2 + ... + a_nX_n) = a_1E(X_1) + a_2E(X_2) + ... + a_nE(X_n)$

  or, using summation notation:

$$E\left(\sum_{i=1}^{n} a_i X_1\right) = \sum_{i=1}^{n} a_i E(X_i)$$

- Then, if each $a_i = 1$:

$$E\left(\sum_{i=1}^{n} 1 \cdot X_1\right) = \sum_{i=1}^{n} 1 \cdot E(X_i) = \sum_{i=1}^{n} E(X_i)$$

The Variance

- The variance of X is a measure of the dispersion of the distribution
- Let $\mu_X = E(X)$. Then the variance can be written as the squared deviations from the mean:

$$\sigma^2 = Var(X) = E\left[(X-\mu)^2\right]$$

- Notice that:

$$E\left[(X-\mu)^2\right] = E(X^2 + \mu^2 - 2X\mu) = E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2$$

- The square root of $Var(X)$ is the standard deviation of X:

$$\sigma_x = sd(X) \equiv +\sqrt{Var(X)}$$

The Variance

- **Var.1** For any constant c: $Var(c) = 0$
- **Var.2** For any constants a and b: $Var(aX + b) = a^2 Var(X)$
- **Var.3** $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- **Var.4** $Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y)$
- **Var. 5** If (and only if) X and Y are independent, then

$$Var(X + Y) = Var(X) + Var(Y)$$

The Covariance

- Concerns the relationship between two variables describing a population
- Let $\mu_X = E(X)$ and $\mu_Y = E(Y)$. Then:

$$\begin{aligned}\sigma_{XY} = Cov(X,Y) &\equiv E\left[(X-\mu_X)(Y-\mu_Y)\right] \\ &= E\left[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y\right] \\ &\text{given that } E(X)=\mu_X \text{ and } E(Y)=\mu_Y\text{:} \\ &= E(XY) - \mu_X\mu_Y\end{aligned}$$

- If $\sigma_{XY} > 0$, on average, when X is above its mean, Y is also above its mean
- If $\sigma_{XY} < 0$, on average, when X is above its mean, Y is below its mean

The Covariance: Properties

- **Cov.1** If X and Y are independent, then $Cov(X, Y) = 0$
- **Cov.2** For any constants a_1, b_1, a_2 and b_2:

$$Cov(a_1 X + b_1, a_2 Y + b_2) = a_1 a_2 Cov(X, Y)$$

- **Cov.3** *Cauchy-Schwartz Inequality*:

$$|Cov(X, Y)| \leq sd(X) sd(Y)$$

The Correlation Coefficient

- The covariance between two random variables depends on the units of measurement
- The correlation coefficient overcomes this issue:

$$\rho_{XY} = Corr(X, Y) \equiv \frac{Cov(X, Y)}{sd(X)sd(Y)} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- Since σ_X and σ_Y are positive, the $Cov(X, Y)$ and $Corr(X, Y)$ have the same sign
- $Corr(X, Y) = 0$ if and only if $Cov(X, Y) = 0$

The Correlation Coefficient: Properties

- **Corr.1** $-1 \leq Corr(X,Y) \leq 1$
  - If $Corr(X,Y) = Cov(X,Y) = 0$ then X and Y are linearly uncorrelated
  - If $Corr(X,Y) = 1$ then X and Y are perfectly positively correlated
  - If $Corr(X,Y) = -1$ then X and Y are perfectly negatively correlated
- **Corr.2** For any constants a_1, b_1, a_2 and b_2:
  - If $a_1 a_2 > 0$:

$$Corr(a_1 X + b_1, a_2 Y + b_2) = Corr(X,Y)$$

  - If $a_1 a_2 < 0$:

$$Corr(a_1 X + b_1, a_2 Y + b_2) = -Corr(X,Y)$$

Variance of Sums of Random Variables

- For any constants a and b:

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

- If X and Y are uncorrelated, i.e., $Cov(X, Y) = 0$, then:

$$Var(X + Y) = Var(X) + Var(Y)$$

and

$$Var(X - Y) = Var(X) + Var(Y)$$

Variance of Sums of Random Variables

- But we can extend these expressions to more than two random variables
- If $\{X_1, ..., X_n\}$ are pairwise uncorrelated random variables and $\{a_1, ..., a_n\}$ are constants, then:

$$Var(a_1 X_1 + a_n X_n) = a_1^2 Var(X_1) + ... + a_n^2 Var(X_n)$$

- Therefore, in summation notation, we have:

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Conditional Expectation and Variation

- Main idea: Suppose we know that $X = x$, then we can compute the expected value Y given that we know that $X = x$
- Let's denote this expected value by $E\left[Y|X = x\right]$
- **Properties:**
  - **CE.1** For any function $c(X)$: $E\left[c(X)|X\right] = c(X)$
  - **CE.2** For any functions $a(X)$ and $b(X)$:

$$E\left[a(X)Y + b(X)|X\right] = a(X)E(Y|X) + b(X)$$

  - **CE.3** If X and Y are independent, then $E(Y|X) = E(Y)$

Conditional Expectation and Variation

- **Properties (cont.):**
  - **CE.4** $E\left[E(Y|X)\right] = E(Y)$
  - **CE.4'** $E(Y|X) = E\left[E(Y|X,Z)|X\right]$
  - **CE.5** If $E(Y|X) = E(Y)$ then: $Cov(X,Y) = Corr(X,Y) = 0$

- The variance of Y conditional on $X = x$ is given by:

$$Var(Y|X = x) = E(Y^2|X = x) - \left[E(Y|x)^2\right]$$

- **Property:**
  - **CV1** If X and Y are independent then: $Var(Y|X) = Var(Y)$