Economics Fundamentals

Lecture 7: Marginal Rate of Substitution, Utility Function, and Utility Maximization

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Recap: Building the Consumer Model

Lecture	Question	Tool
5	What can the consumer **afford**?	Budget constraint: $p_1x_1 + p_2x_2 \leq M$
6	How does the consumer **rank** bundles?	Preferences, axioms, indifference curves
7	**How does the consumer choose?**	**MRS, utility, optimization**

Today we put it all together! 🧩

Today's Roadmap 🧭

Part 1 — Marginal Rate of Substitution

- Definition and intuition
- MRS along an indifference curve
- Diminishing MRS

Part 2 — Utility Functions

- From preferences to utility
- Marginal utility
- MRS = ratio of marginal utilities

Part 3 — Consumer's Optimal Choice

- The tangency condition: $MRS = \frac{p_1}{p_2}$
- Graphical solution
- Algebraic solution (step by step)

Part 4 — Worked Examples

- Tourism applications
- Corner solutions

Part 1: The Marginal Rate of Substitution

MRS: The Intuition 🤔

We know from Lecture 6 that indifference curves are **downward sloping** — to stay equally happy, getting more of one good requires giving up some of the other.

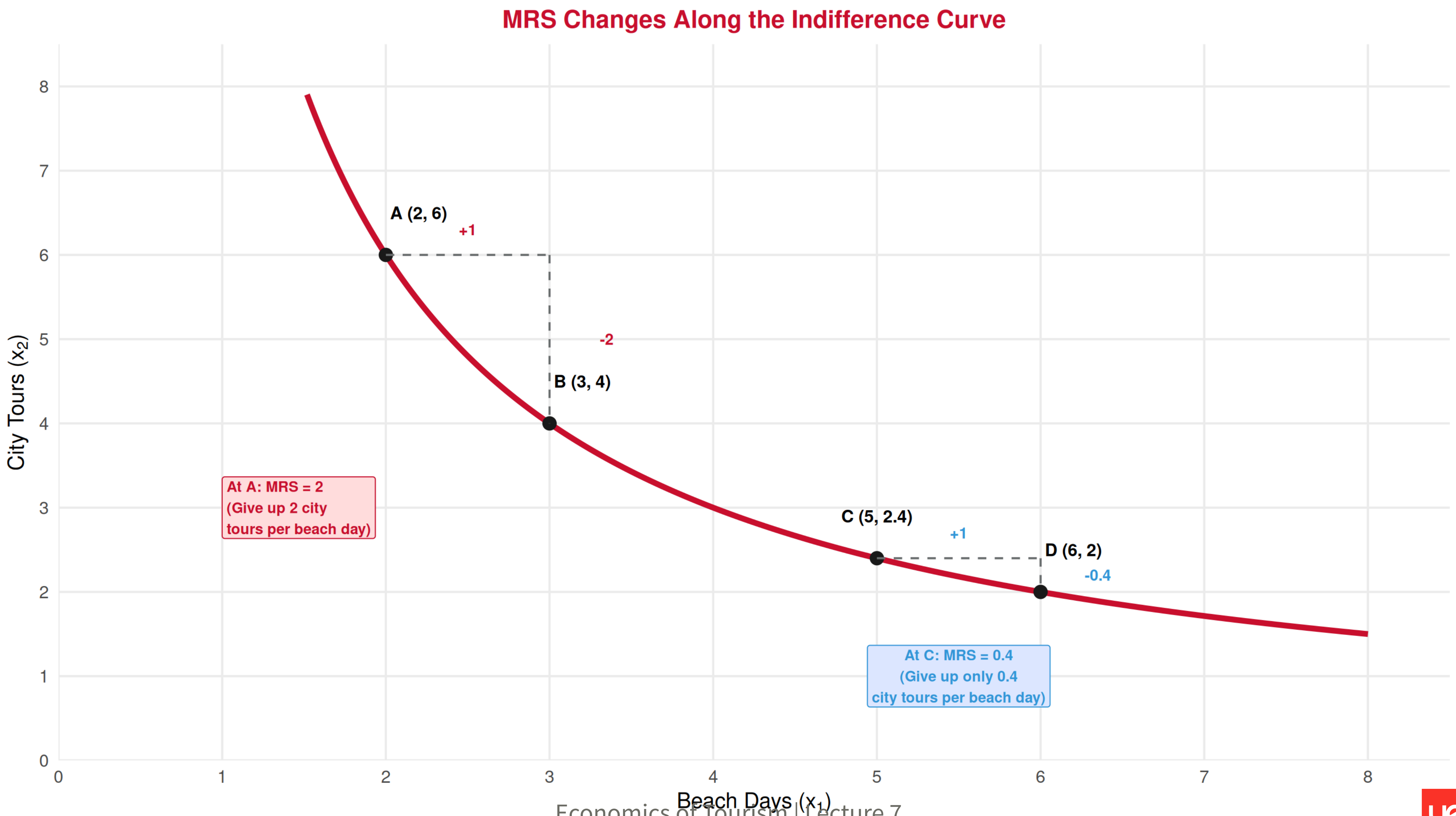
The key question: How much of good 2 is the consumer **willing to give up** to get one more unit of good 1?

MARGINAL RATE OF SUBSTITUTION (MRS)

The MRS measures the rate at which a consumer is willing to **trade good 2 for good 1** while remaining on the same indifference curve.

$$MRS = -\frac{\Delta x_2}{\Delta x_1}\Big|_{\text{along IC}} = \text{slope of the IC (in absolute value)}$$

MRS: Tourism Example 🌴



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Diminishing MRS 📉

LAW OF DIMINISHING MRS

As a consumer has **more** of good 1 and **less** of good 2, they are willing to give up **less** of good 2 for an additional unit of good 1.

Tourism intuition: Imagine you have **1 beach day and 8 city tours**. You'd happily trade several city tours for another beach day.

But if you have **7 beach days and 1 city tour**, you'd need a lot of beach days to compensate for losing that last city tour!

👉 This is what makes indifference curves **convex** (bowed toward the origin).

👉 Diminishing MRS reflects the idea that consumers **prefer variety**.

Part 2: Utility Functions

From Preferences to Utility 📊

In Lecture 6, we described preferences using the symbols $\succ, \sim, \succsim$.

A **utility function** translates those preferences into **numbers**.

UTILITY FUNCTION

A function $U(x_1, x_2)$ that assigns a number to each bundle such that:

$$A \succsim B \iff U(A) \geq U(B)$$

The consumer prefers the bundle with the **higher utility number**.

Important: Utility is **ordinal**, not cardinal. Only the **ranking** matters, not the size of the numbers. If $U(A) = 10$ and $U(B) = 5$, we know $A \succ B$, but we **cannot** say "A is twice as good."

Common Utility Functions ⚙️

Utility Function	Formula	IC Shape	Example
Cobb-Douglas	$U = x_1^a \cdot x_2^b$	Standard (convex)	Beach days & city tours
Perfect Substitutes	$U = ax_1 + bx_2$	Straight lines	Two equivalent airlines
Perfect Complements	$U = \min(ax_1, bx_2)$	L-shaped	Surfboard + accommodation

For this course, we will mostly work with **Cobb-Douglas** preferences.

Example: $U(x_1, x_2) = x_1 \cdot x_2$

Bundle	x_1	x_2	$U = x_1 \cdot x_2$	Ranking
A	2	6	12	$A \sim B$
B	3	4	12	$A \sim B$
C	4	5	20	$C \succ A$

Marginal Utility 🔬

MARGINAL UTILITY (MU)

The additional utility from consuming **one more unit** of a good, holding the other good constant.

$$MU_1 = \frac{\partial U}{\partial x_1} \qquad MU_2 = \frac{\partial U}{\partial x_2}$$

Example: For $U(x_1, x_2) = \sqrt{x_1 \cdot x_2}$:

$$MU_1 = \frac{\partial(\sqrt{x_1 \cdot x_2})}{\partial x_1} = \frac{1}{2}\sqrt{\frac{x_2}{x_1}} \qquad MU_2 = \frac{\partial(\sqrt{x_1 \cdot x_2})}{\partial x_2} = \frac{1}{2}\sqrt{\frac{x_1}{x_2}}$$

👉 **Diminishing marginal utility**: As you consume more of a good, each additional unit adds less satisfaction (same idea from the ice cream cone example in the Lecture Notes availabe in Canvas.).

The Key Link: MRS = Ratio of Marginal Utilities 🔑

There is a powerful connection between the MRS (from indifference curves) and marginal utilities (from the utility function):

MRS AND MARGINAL UTILITY

$$MRS = \frac{MU_1}{MU_2}$$

The MRS equals the ratio of marginal utilities of the two goods.

Intuition: If good 1 gives you a lot of extra utility (MU_1 is high) and good 2 gives you little (MU_2 is low), you are willing to give up a lot of good 2 for one more unit of good 1 — so MRS is high.

Example: For $U = x_1 \cdot x_2$ at bundle $(2, 6)$:

$$MRS = \frac{MU_1}{MU_2} = \frac{x_2}{x_1} = \frac{6}{2} = 3$$

👉 The consumer would give up **3 units of good 2** for **1 more unit of good 1**.

Why Does MRS = MU_1 / MU_2? 💡

Derivation (along an indifference curve, utility stays constant):

$$dU = 0$$

$$\frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

$$MU_1 \cdot dx_1 + MU_2 \cdot dx_2 = 0$$

Rearranging:

$$-\frac{dx_2}{dx_1} = \frac{MU_1}{MU_2}$$

$$\boxed{MRS = \frac{MU_1}{MU_2}}$$

👉 This formula lets us compute the MRS directly from the utility function — no graph needed!

Part 3: The Consumer’s Optimal Choice

The Consumer's Problem 🎯

Now we can put **everything together**:

THE CONSUMER'S PROBLEM

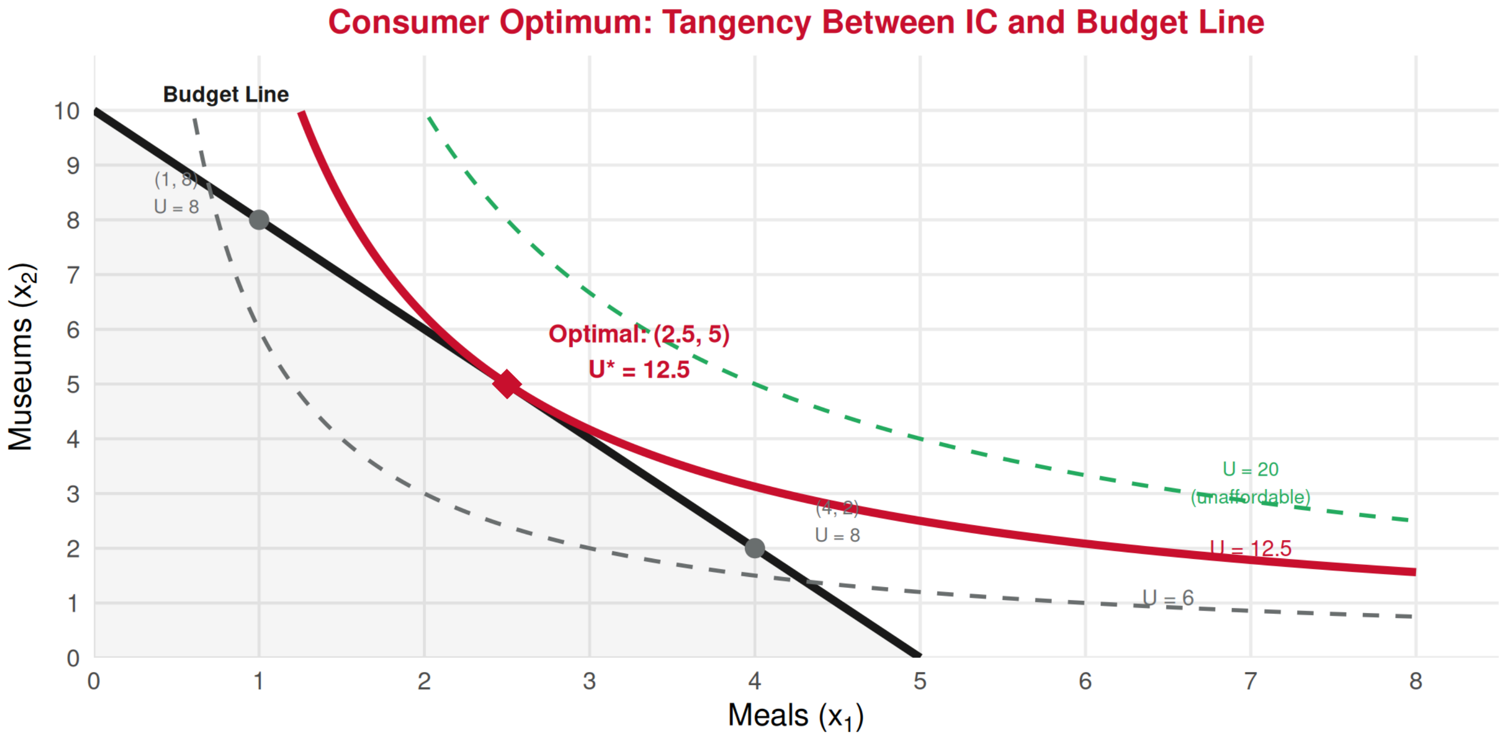
Maximize utility subject to the budget constraint:

$$\max_{x_1, x_2} U(x_1, x_2) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = M$$

In words: Choose the bundle on the **highest possible indifference curve** that is still **affordable** (on or below the budget line).

Since more is better (non-satiation), the consumer will always spend all income ➡️ the optimal bundle is **on** the budget line.

Graphical Solution: Tangency 🎨



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The Tangency Condition 📏

At the optimum, the **slope of the IC** equals the **slope of the budget line**:

OPTIMALITY CONDITION (INTERIOR SOLUTION)

$$MRS = \frac{p_1}{p_2}$$

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

Or equivalently:

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

Interpretation of $\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$: The **marginal utility per euro** spent must be equal for both goods. If it is not, the consumer can improve by reallocating spending.

Why Must MRS = Price Ratio? 💡

If MRS > p_1/p_2 ➡️

The consumer values good 1 **more** than the market does.

The consumer should **buy more** of good 1 and **less** of good 2.

This moves them **down along the budget line**, increasing utility.

If MRS < p_1/p_2 ⬅️

The consumer values good 1 **less** than the market does.

The consumer should **buy less** of good 1 and **more** of good 2.

This moves them **up along the budget line**, increasing utility.

Only when MRS = p_1/p_2 is there no room for improvement — the consumer is at the **optimum**!

Analogy from Lecture 3: This is exactly the **cost-benefit principle** applied to marginal reallocation of spending.

Solving Algebraically: Step by Step 🧮

The method (2 equations, 2 unknowns):

Equation 1 — Tangency condition:

$$MRS = \frac{p_1}{p_2} \implies \frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

Equation 2 — Budget constraint:

$$p_1 x_1 + p_2 x_2 = M$$

Steps:

1️⃣ Compute MU_1 and MU_2 from the utility function

2️⃣ Set $MRS = p_1/p_2$ and solve for x_2 in terms of x_1 (or vice versa)

3️⃣ Substitute into the budget constraint

4️⃣ Solve for x_1^* and x_2^*

Worked Example: Tourist in Lisbon 🇵🇹

Problem: $U(x_1, x_2) = x_1 \cdot x_2$, with $p_1 = 20, p_2 = 10, M = 100.$

Step 1: Marginal utilities

$$MU_1 = \frac{\partial(x_1 x_2)}{\partial x_1} = x_2 \qquad MU_2 = \frac{\partial(x_1 x_2)}{\partial x_2} = x_1$$

Step 2: Tangency condition

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2} \implies \frac{x_2}{x_1} = \frac{20}{10} = 2 \implies x_2 = 2x_1$$

Step 3: Substitute into budget constraint

$$20x_1 + 10(2x_1) = 100 \implies 20x_1 + 20x_1 = 100 \implies 40x_1 = 100$$

Step 4: Solve

$$\boxed{x_1^* = 2.5 \text{ meals}} \qquad x_2^* = 2(2.5) = \boxed{5 \text{ museums}}$$

$$U^* = 2.5 \times 5 = 12.5$$

Verifying: $MU_1/p_1 = MU_2/p_2$ ✅

At the optimum $(x_1^*, x_2^*) = (2.5, 5)$:

$$\frac{MU_1}{p_1} = \frac{x_2}{p_1} = \frac{5}{20} = 0.25 \text{ utils per euro}$$

$$\frac{MU_2}{p_2} = \frac{x_1}{p_2} = \frac{2.5}{10} = 0.25 \text{ utils per euro}$$

✅ Equal! The last euro spent on meals gives the **same** additional satisfaction as the last euro spent on museums.

👉 If $MU_1/p_1 > MU_2/p_2$, the consumer should shift spending toward good 1 (meals give more "bang for the buck").

👉 If $MU_1/p_1 < MU_2/p_2$, shift toward good 2.

Another Example: Cobb-Douglas $U = x_1^{0.4} x_2^{0.6}$ ✏️

Problem: $p_1 = 5, p_2 = 10, M = 200.$

Step 1: $MU_1 = 0.4 \cdot x_1^{-0.6} \cdot x_2^{0.6} \qquad MU_2 = 0.6 \cdot x_1^{0.4} \cdot x_2^{-0.4}$

Step 2: Tangency

$$\frac{MU_1}{MU_2} = \frac{0.4\, x_2^{0.6}\, x_1^{-0.6}}{0.6\, x_1^{0.4}\, x_2^{-0.4}} = \frac{0.4}{0.6} \cdot \frac{x_2}{x_1} = \frac{2}{3} \cdot \frac{x_2}{x_1}$$

Setting equal to $p_1/p_2 = 5/10 = 1/2$:

$$\frac{2}{3} \cdot \frac{x_2}{x_1} = \frac{1}{2} \implies x_2 = \frac{3}{4} x_1$$

Step 3: Budget constraint: $5x_1 + 10\left(\frac{3}{4}x_1\right) = 200 \implies 5x_1 + 7.5x_1 = 200 \implies 12.5x_1 = 200$

Step 4: $\boxed{x_1^* = 16} \qquad x_2^* = \frac{3}{4}(16) = \boxed{12}$

💡 **Shortcut for Cobb-Douglas $U = x_1^a x_2^b$**: spend fraction $\frac{a}{a+b}$ of income on good 1, and $\frac{b}{a+b}$ on good 2!

The Cobb-Douglas Shortcut 🚀

COBB-DOUGLAS DEMAND SHORTCUT

For $U = x_1^a \cdot x_2^b$, the optimal demands are:

$$x_1^* = \frac{a}{a+b} \cdot \frac{M}{p_1} \qquad x_2^* = \frac{b}{a+b} \cdot \frac{M}{p_2}$$

Verify with our previous example: $a = 0.4, b = 0.6, M = 200, p_1 = 5, p_2 = 10$

$$x_1^* = \frac{0.4}{1} \cdot \frac{200}{5} = 0.4 \times 40 = 16 \quad \checkmark$$

$$x_2^* = \frac{0.6}{1} \cdot \frac{200}{10} = 0.6 \times 20 = 12 \quad \checkmark$$

👉 This shortcut works for **any** Cobb-Douglas utility. The **exponents determine budget shares**.

Part 4: Special Cases and Intuition

Corner Solutions 🚩

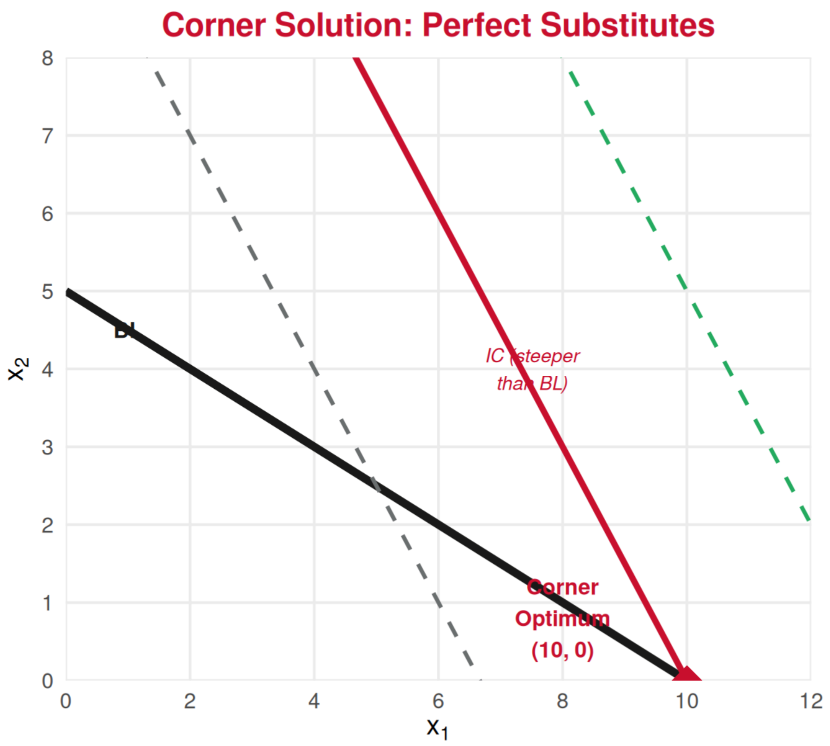
The tangency condition gives **interior solutions** (positive amounts of both goods).

Sometimes the optimum is at a **corner** — the consumer buys **only one good**.

This happens with **perfect substitutes** when $MRS \neq p_1/p_2$ everywhere.

Rule for perfect substitutes $U = ax_1 + bx_2$:

- If $\frac{a}{b} > \frac{p_1}{p_2}$: buy only good 1
- If $\frac{a}{b} < \frac{p_1}{p_2}$: buy only good 2
- If $\frac{a}{b} = \frac{p_1}{p_2}$: any bundle on BL is optimal



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Perfect Complements: Optimum at the Kink 🧩

For **perfect complements** $U = \min(ax_1, bx_2)$:

No tangency exists (the IC has a kink!). The optimum is always at the **corner of the L**.

Condition: $ax_1 = bx_2$

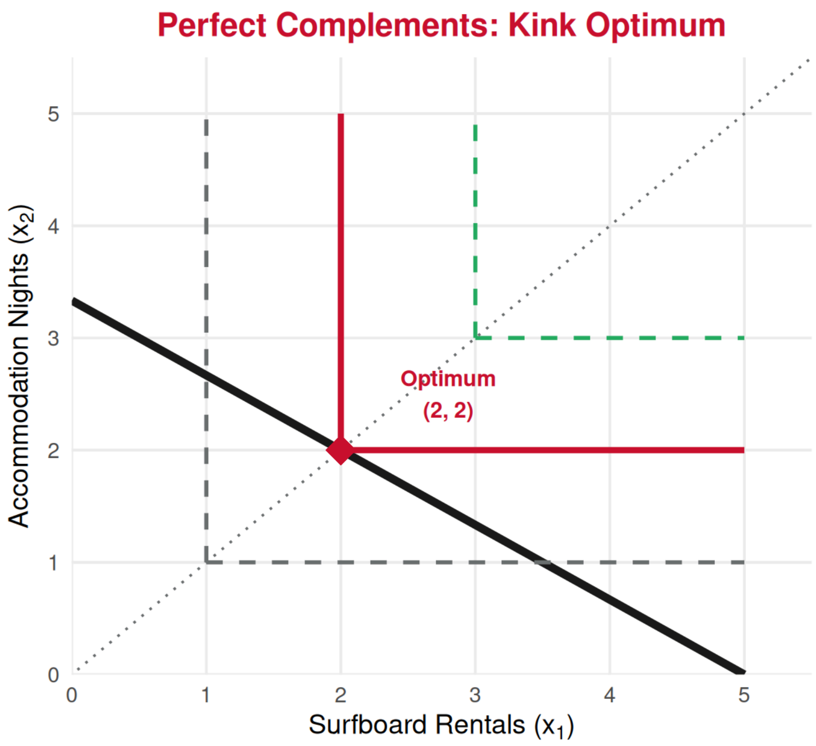
Substitute into the budget constraint:

$$p_1x_1 + p_2\left(\frac{a}{b}x_1\right) = M$$

Example: "Surf & Stay" package, $U = \min(x_1, x_2)$, $p_1 = 40$, $p_2 = 60$, $M = 200$:

$$40x_1 + 60x_1 = 200$$

$$x_1^* = x_2^* = 2$$



Summary of the Consumer’s Solution 📋

Preference Type	Utility Function	Solution Method	Optimal Condition
Standard (Cobb-Douglas)	$x_1^a x_2^b$	Tangency + Budget	$MRS = p_1/p_2$
Perfect Substitutes	$ax_1 + bx_2$	Compare $\frac{a}{b}$ vs $\frac{p_1}{p_2}$	Corner or entire BL
Perfect Complements	$\min(ax_1, bx_2)$	Kink + Budget	$ax_1 = bx_2$

THE CONSUMER’S OPTIMAL CHOICE — MASTER SUMMARY

1️⃣ Write the utility function and compute MU_1, MU_2

2️⃣ Set $\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$ (tangency)

3️⃣ Combine with $p_1 x_1 + p_2 x_2 = M$ (budget)

4️⃣ Solve the system for x_1^* and x_2^*

Summary: Today’s Key Takeaways

1️⃣ **MRS** = rate at which the consumer trades good 2 for good 1 along an IC

2️⃣ **Diminishing MRS** ➡️ indifference curves are convex ➡️ consumers prefer variety

3️⃣ **Utility functions** assign numbers to bundles; $MRS = MU_1/MU_2$

4️⃣ **Optimal choice**: highest IC touching the budget line ➡️ tangency condition $MRS = p_1/p_2$

5️⃣ **Equivalent condition**: $MU_1/p_1 = MU_2/p_2$ (equal marginal utility per euro)

6️⃣ **Cobb-Douglas shortcut**: spend fraction $a/(a+b)$ on good 1

Next lecture: Lecture 8 — From individual demand to **market demand** and **linear demand curves**.

Exercises

Application Time! ✏️

MRS, utility maximization, and graphical analysis.

Exercise 1: Multiple Choice

Question: A consumer has utility $U = x_1 \cdot x_2$ and currently consumes the bundle $(4, 8)$. The MRS at this point is:

A. 32

B. 4

C. 2

D. 0.5

Answer: C — MRS = 2

$MRS = \frac{MU_1}{MU_2} = \frac{x_2}{x_1} = \frac{8}{4} = 2$. The consumer is willing to give up 2 units of good 2 for 1 additional unit of good 1.

Exercise 2: Multiple Choice

Question: At the optimal bundle, if $MU_1/p_1 > MU_2/p_2$, the consumer should:

A. Buy more of good 1 and less of good 2

B. Stay at the current bundle — it is already optimal

C. Buy more of good 2 and less of good 1

D. Increase total spending

Answer: A — Buy more of good 1 and less of good 2

If $MU_1/p_1 > MU_2/p_2$, the last euro spent on good 1 gives more satisfaction than the last euro on good 2. Shifting spending toward good 1 increases total utility. The consumer continues until $MU_1/p_1 = MU_2/p_2$.

Exercise 3: Open Question

A tourist in the Algarve has a daily budget of €120 to split between boat tours (x_1, price €30 each) and restaurant meals (x_2, price €20 each). Their utility function is $U(x_1, x_2) = x_1^{0.5} \cdot x_2^{0.5}$.

a. Write the budget constraint equation and find the intercepts.

b. Compute MU_1 and MU_2. Derive the MRS.

c. Using the tangency condition ($MRS = p_1/p_2$) and the budget constraint, find the optimal bundle (x_1^*, x_2^*).

d. Verify your answer using the Cobb-Douglas shortcut ($a = b = 0.5$).

e. Compute the utility at the optimum. Now suppose the tourist's budget increases to €180 (everything else unchanged). Find the new optimal bundle and new utility. By what percentage did utility increase?

Hint: For part (b), recall that the partial derivative of $x^{0.5}$ is $0.5x^{-0.5}$. For part (d), the Cobb-Douglas shortcut says spend fraction $\frac{a}{a+b}$ on each good.

Exercise 3: Solution

a. $30x_1 + 20x_2 = 120, x_2 = 6 - 1.5x_1$. When $x_1 = 0, x_2 = 6$, when $x_2 = 0, x_1 = 4$.

b. $MU_1 = \frac{1}{2}\sqrt{x_2/x_1}, MU_2 = \frac{1}{2}\sqrt{x_1/x_2}, MRS = x_2/x_1$

c. $(x_1^*, x_2^*) = (2, 3)$

d. ✅

e. $u(2,3) = \sqrt{2 \cdot 3} = \sqrt{6} \approx 2.45$. If $M = 180, (x_1^*, x_2^*) = (3, 4.5)$,
$u(3, 4.5) = \sqrt{3 \cdot 4.5} = \sqrt{13.5} \approx 3.67$. The utility increased $\frac{3.67}{2.45} \approx 1.5$, that is about 50%.

Next Lecture

February 27, 2026: Demand — Individual and Market Demand, Linear Demands

Thank You!

Questions?

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Next class: Friday, February 27, 2026